

A geometric construction of Coxeter-Dynkin diagrams of bimodal singularities

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We consider the Berglund-Hübsch transpose of a bimodal invertible polynomial and construct a triangulated category associated to the compactification of a suitable deformation of the singularity. This is done in such a way that the corresponding Grothendieck group with the (negative) Euler form can be described by a graph which corresponds to the Coxeter-Dynkin diagram with respect to a distinguished basis of vanishing cycles of the bimodal singularity.

Introduction

Let $f(x, y, z)$ be a weighted homogeneous polynomial which has an isolated singularity at the origin $0 \in \mathbb{C}^3$. An important invariant of f is a Coxeter-Dynkin diagram with respect to a distinguished basis of vanishing cycles in the Milnor fibre of f . It determines the monodromy of the singularity as the corresponding Coxeter element. The vanishing cycles can be chosen to be (graded) Lagrangian submanifolds of the Milnor fibre. A distinguished basis of such vanishing Lagrangian cycles can be categorified to an A_∞ -category $\mathrm{Fuk}^\rightarrow(f)$ called the directed Fukaya category of f . Its derived category $D^b\mathrm{Fuk}^\rightarrow(f)$ is, as a triangulated category, an invariant of the polynomial f .

On the other hand, one can consider the bounded derived category of coherent sheaves on a resolution of the singularity or of a compactification of the Milnor fibre as in [EP1]. The homological mirror symmetry conjecture states that there should be a relation between these categories for mirror symmetric singularities.

In [ET], the first author and A. Takahashi considered a mirror symmetry in a specific class of weighted homogeneous polynomials in three variables, namely the so called invertible polynomials. The mirror symmetry is given by the Berglund-Hübsch transpose f^T of f . They generalised Arnold's strange duality for the 14 exceptional unimodal singularities to this wider class. They defined Dolgachev and Gabrielov numbers for

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such invertible polynomials and showed that the Dolgachev numbers of f coincide with the Gabrielov numbers of f^T and the Gabrielov numbers of f coincide with the Dolgachev numbers of f^T .

In the case of the 14 exceptional unimodal singularities, the Gabrielov numbers are directly related with a Coxeter-Dynkin diagram of the singularity. In [EP1], it was shown that one can find a Coxeter-Dynkin diagram of the dual singularity in the bounded derived category of coherent sheaves on a resolution of the compactification of the Milnor fibre of f .

In this paper, we consider the bimodal singularities. They were also classified by V. I. Arnold. They fall into 8 infinite series starting with 6 classes where, setting one modulus equal to 0, one obtains weighted homogeneous polynomials. Besides these series, there are again 14 exceptional singularities. In these 6+14 classes one finds invertible polynomials. Coxeter-Dynkin diagrams for the bimodal singularities were computed in [Eb]. In this paper, we shall show that these Coxeter-Dynkin diagrams can be constructed geometrically in a way similar to [EP1] using suitable invertible polynomials and their Berglund-Hübsch transposes.

1 Invertible polynomials

Let $f(x_1, \dots, x_n)$ be a weighted homogeneous complex polynomial. This means that there are positive integers w_1, \dots, w_n and d such that

$$f(\lambda^{w_1}x_1, \dots, \lambda^{w_n}x_n) = \lambda^d f(x_1, \dots, x_n)$$

for $\lambda \in \mathbb{C}^*$. We call $(w_1, \dots, w_n; d)$ a system of *weights*. The weight system is said to be *reduced* if $\gcd(w_1, \dots, w_n, d) = 1$; otherwise it is called *non-reduced*. Recall that a quasihomogeneous polynomial $f(x_1, \dots, x_n)$ in n variables is called *invertible* if it is of the form

$$f(x_1, \dots, x_n) = \sum_{i=1}^n a_i \prod_{j=1}^n x_j^{E_{ij}}$$

for some coefficients $a_i \in \mathbb{C}^*$ and for a matrix $E = (E_{ij})$ with non-negative integer entries and with $\det E \neq 0$. For simplicity we can assume $a_i = 1$ for $i = 1, \dots, n$. (This can be achieved by a suitable rescaling of the variables.) An invertible quasihomogeneous polynomial f is called *non-degenerate* if it has (at most) an isolated critical point at the origin in \mathbb{C}^n . An invertible polynomial has a *canonical system of weights*: This is the system of weights $W_f = (w_1, \dots, w_n; d')$ given by the unique solution of the equation

$$E \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \det(E) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad d' := \det(E).$$

This system of weights is in general non-reduced. Define

$$c_f := \gcd(w_1, \dots, w_n, d').$$

Let

$$(q_1, \dots, q_n; d) := (w_1/c_f, \dots, w_n/c_f; d'/c_f)$$

be the corresponding reduced weight system. We define the *Berglund-Hübsch transpose* $f^T(x_1, \dots, x_n)$ of an invertible polynomial $f(x_1, \dots, x_n)$ by

$$f^T(x_1, \dots, x_n) := \sum_{i=1}^n a_i \prod_{j=1}^n x_j^{E_{ji}}.$$

2 Weighted homogeneous bimodal singularities

The bimodal singularities have been classified by Arnold [A1, A2]. They are characterised by the fact that the exceptional divisor of the minimal resolution is a Kodaira degenerate elliptic curve of type I_p^* , $p \geq 0$, IV^* , III^* , or II^* with a different neighbourhood [Ku, EW]. In the classes I_0^* , IV^* , III^* , and II^* one can find weighted homogeneous polynomials. The list of classes with the names given by Arnold and their deformations is given in Table 1. We also indicate the number r of components of the exceptional divisor with a self-intersection number different from -2 .

The 6 singularities of Kodaira type I_0^* are referred to as *quadrilateral singularities* since they correspond to certain quadrangles in the hyperbolic plane in the same way as the 14 exceptional unimodal singularities correspond to triangles in the hyperbolic plane [D1]. The remaining singularities of Kodaira types IV^* , III^* , II^* are called *exceptional*.

r	I_0^*		IV^*		III^*		II^*
1	$J_{3,0}$	\leftarrow	E_{18}	\leftarrow	E_{19}	\leftarrow	E_{20}
1	$Z_{1,0}$	\leftarrow	Z_{17}	\leftarrow	Z_{18}	\leftarrow	Z_{19}
1	$Q_{2,0}$	\leftarrow	Q_{16}	\leftarrow	Q_{17}	\leftarrow	Q_{18}
2	$W_{1,0}$	\leftarrow	W_{17}	\leftarrow	W_{18}		
2	$S_{1,0}$	\leftarrow	S_{16}	\leftarrow	S_{17}		
3	$U_{1,0}$	\leftarrow	U_{16}				

Table 1: Weighted homogeneous bimodal singularities

In fact, in each of these classes one can find non-degenerate invertible polynomials in three variables. In Table 2 there are chosen invertible polynomials for these classes and in each case, the Berglund-Hübsch transpose is indicated. Since the Berglund-Hübsch transpose will be our main concern, we shall denote it by f and the invertible polynomial for the bimodal singularity by f^T . We also indicate the Dolgachev numbers $\alpha_1, \alpha_2, \alpha_3$ and Gabrielov numbers $\gamma_1, \gamma_2, \gamma_3$ for f as defined in [ET]. They are the Gabrielov numbers and Dolgachev numbers of the polynomial f^T respectively by [ET]. Note that these numbers depend on the polynomial f and, in general, they differ from the Dolgachev numbers of the singularity in [D1].

Name	$\gamma_1, \gamma_2, \gamma_3$	f^T	f	$\alpha_1, \alpha_2, \alpha_3$	Dual
$J_{3,0}$	2, 4, 6	$x^6y + y^3 + z^2$	$x^6 + xy^3 + z^2$	2, 3, 10	Z_{13}
$Z_{1,0}$	2, 4, 8	$x^5y + xy^3 + z^2$	$x^5y + xy^3 + z^2$	2, 4, 8	$Z_{1,0}$
$Q_{2,0}$	2, 4, 10	$x^4y + y^3 + xz^2$	$x^4z + xy^3 + z^2$	3, 3, 7	Z_{17}
$W_{1,0}$	2, 6, 6	$x^6 + y^2 + yz^2$	$x^6 + y^2z + z^2$	2, 6, 6	$W_{1,0}$
$S_{1,0}$	2, 6, 8	$x^5 + xy^2 + yz^2$	$x^5y + y^2z + z^2$	3, 5, 5	W_{17}
$U_{1,0}$	3, 4, 6	$x^3 + xy^2 + yz^3$	$x^3y + y^2z + z^3$	3, 4, 6	$U_{1,0}$
E_{18}	3, 3, 5	$x^5z + y^3 + z^2$	$x^5 + y^3 + xz^2$	2, 3, 12	Q_{12}
E_{19}	2, 4, 7	$x^7y + y^3 + z^2$	$x^7 + xy^3 + z^2$	2, 3, 12	$Z_{1,0}$
E_{20}	2, 3, 11	$x^{11} + y^3 + z^2$	$x^{11} + y^3 + z^2$	2, 3, 11	E_{20}
Z_{17}	3, 3, 7	$x^4z + xy^3 + z^2$	$x^4y + y^3 + xz^2$	2, 4, 10	$Q_{2,0}$
Z_{18}	2, 4, 10	$x^6y + xy^3 + z^2$	$x^6y + xy^3 + z^2$	2, 4, 10	Z_{18}
Z_{19}	2, 3, 16	$x^9 + xy^3 + z^2$	$x^9y + y^3 + z^2$	2, 4, 9	E_{25}
Q_{16}	3, 3, 9	$x^4z + y^3 + xz^2$	$x^4z + y^3 + xz^2$	3, 3, 9	Q_{16}
Q_{17}	2, 4, 13	$x^5y + y^3 + xz^2$	$x^5z + xy^3 + z^2$	3, 3, 9	$Z_{2,0}$
Q_{18}	2, 3, 21	$x^8 + y^3 + xz^2$	$x^8z + y^3 + z^2$	3, 3, 8	E_{30}
W_{17}	3, 5, 5	$x^5z + yz^2 + y^2$	$x^5 + xz^2 + y^2z$	2, 6, 8	$S_{1,0}$
W_{18}	2, 7, 7	$x^7 + y^2 + yz^2$	$x^7 + y^2z + z^2$	2, 7, 7	W_{18}
S_{16}	3, 5, 7	$x^4y + xz^2 + y^2z$	$x^4y + xz^2 + y^2z$	3, 5, 7	S_{16}
S_{17}	2, 7, 10	$x^6 + xy^2 + yz^2$	$x^6y + y^2z + z^2$	3, 6, 6	$X_{2,0}$
U_{16}	5, 5, 5	$x^5 + y^2z + yz^2$	$x^5 + y^2z + yz^2$	5, 5, 5	U_{16}

Table 2: Strange duality of the bimodal singularities

In each case, the invertible polynomial f defines another singularity whose name (in Arnold's notation) is also given in the table. Note we have chosen two invertible polynomials in the singularity class $Z_{1,0}$ whose Berglund-Hübsch transposes lie in different classes of bimodal singularities, namely $Z_{1,0}$ and E_{19} .

Coxeter-Dynkin diagrams with respect to distinguished bases of vanishing cycles for these singularities were determined in [Eb]. By a Coxeter-Dynkin diagram we mean the following graph. Let $(L, \langle -, - \rangle)$ be an integral lattice, i.e. L is a finitely generated free \mathbb{Z} -module equipped with a symmetric bilinear form $\langle -, - \rangle$ with values in \mathbb{Z} . An element $e \in L$ with $\langle e, e \rangle = -2$ is called a *root*. Such an element e defines a reflection

$$s_e(x) = x - \frac{2\langle x, e \rangle}{\langle e, e \rangle} = x + \langle x, e \rangle e \text{ for } x \in L.$$

Let $B = (e_1, \dots, e_n)$ be a basis of L consisting of roots. The symmetric bilinear form $\langle -, - \rangle$ with respect to this ordered basis is encoded by a graph, the so called *Coxeter-Dynkin diagram* corresponding to the basis B , in the following way: The vertices correspond to the basis elements e_i and two vertices e_i and e_j with $i \neq j$ are joined by

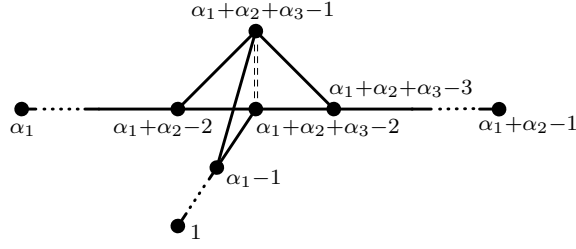


Figure 1: The graph $T(\alpha_1, \alpha_2, \alpha_3)$

$|\langle e_i, e_j \rangle|$ edges which are dashed if $\langle e_i, e_j \rangle < 0$. The *Coxeter element* τ corresponding to B is defined by

$$\tau = s_{e_1} s_{e_2} \cdots s_{e_n}.$$

In the singularity case, we are interested in the Milnor lattice L and a Coxeter-Dynkin diagram corresponding to a distinguished basis of vanishing cycles of the Milnor lattice. Then the Coxeter element corresponding to such a basis is the monodromy operator of the singularity.

According to [Eb] (see also [ET]), a Coxeter-Dynkin diagram with respect to a distinguished basis of vanishing cycles of one of the bimodal singularities can be obtained by the following rule from the invariants of Table 3: Here $(\alpha_1, \alpha_2, \alpha_3)$ are the Dolgachev numbers of f . The number a is the *Gorenstein parameter* of the canonical system of weights $W_{f^T} = (w_1^T, w_2^T, w_3^T; d^T)$ of f^T , i.e.

$$a := d^T - w_1^T - w_2^T - w_3^T.$$

Let $T(\alpha_1, \alpha_2, \alpha_3)$ be the T-shaped graph of Figure 1.

- If $a = 2$ then the diagram $T(\alpha_1, \alpha_2, \alpha_3)$ is extended by $\bullet_1 - \bullet_2$ where \bullet_1 is connected to the upper central vertex and \bullet_2 to the $\alpha_i - \beta_i - 1$ -th vertex from the outside of the i -th arm, unless $\beta_i = \alpha_i - 1$ ($i = 1, 2, 3$).
- If $a = 3$ then the diagram $T(\alpha_1, \alpha_2, \alpha_3)$ is extended by $\bullet_1 - \bullet_2 - \bullet_3$ where \bullet_1 is connected to the upper central vertex and \bullet_3 to the $\alpha_i - \beta_i - 1$ -th vertex from the outside of the i -th arm, unless $\beta_i = \alpha_i - 1$ ($i = 1, 2, 3$).
- If $a = 5$ then the diagram $T(\alpha_1, \alpha_2, \alpha_3)$ is extended by $\bullet_1 - \bullet_2 - \bullet_3 - \bullet_4 - \bullet_5$ where \bullet_1 is connected to the upper central vertex and \bullet_3 to the $\alpha_i - \beta_i - 1$ -th vertex from the outside of the i -th arm, unless $\beta_i = \alpha_i - 1$ ($i = 1, 2, 3$).

The numbering of the vertices of the complete graph is obtained by taking the new vertices as last vertices, in their indicated order.

Note that in the cases where the canonical systems of weights of f are reduced ($c_f = 1$), the numbers β_i of Table 3 satisfy $a\beta_i \equiv 1 \pmod{\alpha_i}$, $i = 1, 2, 3$. Therefore, in these cases, the invariants $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)$ are just the orbit invariants of the \mathbb{C}^* -action on the corresponding singularity, by [D3].

Dual	c_f	$(\alpha_i, \beta_i), i = 1, 2, 3$	a	Name
Z_{13}	2	$(2, 1), (3, 2), (10, 7)$	2	$J_{3,0}$
$Z_{1,0}$	2	$(2, 1), (4, 3), (8, 5)$	2	$Z_{1,0}$
Z_{17}	1	$(3, 2), (3, 2), (7, 4)$	2	$Q_{2,0}$
$W_{1,0}$	2	$(2, 1), (6, 4), (6, 4)$	2	$W_{1,0}$
W_{17}	1	$(3, 2), (5, 3), (5, 3)$	2	$S_{1,0}$
$U_{1,0}$	2	$(3, 1), (4, 3), (6, 4)$	2	$U_{1,0}$
Q_{12}	2	$(2, 1), (3, 2), (12, 8)$	2	E_{18}
$Z_{1,0}$	3	$(2, 1), (3, 2), (12, 9)$	3	E_{19}
E_{20}	1	$(2, 1), (3, 2), (11, 9)$	5	E_{20}
$Q_{2,0}$	2	$(2, 1), (4, 3), (10, 6)$	2	Z_{17}
Z_{18}	1	$(2, 1), (4, 3), (10, 7)$	3	Z_{18}
E_{25}	1	$(2, 1), (4, 3), (9, 7)$	5	Z_{19}
Q_{16}	1	$(3, 2), (3, 2), (9, 5)$	2	Q_{16}
$Z_{2,0}$	3	$(3, 2), (3, 2), (9, 6)$	3	Q_{17}
E_{30}	1	$(3, 2), (3, 2), (8, 6)$	5	Q_{18}
$S_{1,0}$	2	$(2, 1), (6, 4), (8, 5)$	2	W_{17}
W_{18}	1	$(2, 1), (7, 5), (7, 5)$	3	W_{18}
S_{16}	1	$(3, 2), (5, 3), (7, 4)$	2	S_{16}
$X_{2,0}$	3	$(3, 2), (6, 4), (6, 4)$	3	S_{17}
U_{16}	1	$(5, 3), (5, 3), (5, 3)$	2	U_{16}

Table 3: Invariants of the singularities

3 Deformations and compactifications

Our aim is to realize such a Coxeter-Dynkin diagram in a geometric way using the resolution of the compactification of a suitable deformation of the singularity $f(x, y, z)$ dual to the given singularity.

We consider one of the invertible polynomials $f(x, y, z)$ of Table 2. Let $(q_1, q_2, q_3; d)$ be the reduced weight system of f . We consider a suitable deformation f_w of f and a compactification of the level set $f_w = 0$ in a weighted projective 3-space. Let

$$q_0 := d - q_1 - q_2 - q_3$$

and consider the weighted projective space $\mathbb{P}(Q) = \mathbb{P}(q_0, q_1, q_2, q_3)$ with homogeneous coordinates $(w : x : y : z)$ (cf. [D2]). In this weighted projective space, we consider the quasismooth (i.e. the affine cone is smooth outside the vertex) hypersurface

$$Z := \{(w : x : y : z) \in \mathbb{P}(q_0, q_1, q_2, q_3) \mid F(w, x, y, z) = 0\},$$

where

$$F(w, x, y, z) = f(x, y, z) + w^{d/q_0}$$

in the case of the quadrilateral singularities and one of

$$F(w, x, y, z) = \begin{cases} f(x, y, z) + zw^{(d-q_3)/q_0} \\ f(x, y, z) + yw^{(d-q_2)/q_0} \\ f(x, y, z) + xw^{(d-q_1)/q_0} \end{cases}$$

in the case of the 14 exceptional bimodal singularities. See Table 4 for the actual choice of deformation and compactification.

By [D2, 3.3.4 Theorem], Z is a simply connected projective surface with trivial dualizing sheaf $\omega_Z = \mathcal{O}_Z$. Let $c := c_f$. If the canonical system of weights is reduced, we set $Y := Z$. Otherwise, we consider an action of the cyclic group $\mathbb{Z}_c = \mathbb{Z}/c\mathbb{Z}$ on $\mathbb{P}(q_0, q_1, q_2, q_3)$, where a generator $\zeta \in \mathbb{Z}_c$ acts as follows

$$(w : x : y : z) \mapsto (\zeta^{m_0} w : \zeta^{m_1} x : \zeta^{m_2} y : \zeta^{m_3} z)$$

and the corresponding quadruples $(m_0, m_1, m_2, m_3) \in \mathbb{Z}^4$ are indicated in Table 4. This action leaves the surface Z invariant. In these cases let $Y := Z/\mathbb{Z}_c$ be the quotient variety.

Proposition 1. *The variety Y is a simply-connected projective surface with the dualizing sheaf $\omega_Y = \mathcal{O}_Y$.*

Proof. Since the surface Z is simply connected, it is clear that the surface Y is still simply connected. Since $\omega_Z = \mathcal{O}_Z$, the space of holomorphic 2-forms on Z is generated by the holomorphic 2-form

$$\omega_0 := \frac{q_0 w dx dy dz - q_1 x dw dy dz + q_2 y dw dx dz - q_3 z dw dx dy}{dF}$$

(cf. [Sa]). It is easy to see that this 2-form is invariant under the action of \mathbb{Z}_c . □

The singularities of Y are cyclic quotient singularities. Let $\pi: X \rightarrow Y$ be a minimal resolution of its singularities. By Proposition 1, X is a smooth K3 surface. We summarise the relation between the three surfaces:

X	smooth K3 surface
\downarrow resolution π	
$Y = Z/\mathbb{Z}_c$	
\uparrow covering	
$Z = V(F)$	hypersurface in weighted projective space; compactification of Bergland-Hübsch dual of a bimodal singularity

Dual	$F(w, x, y, z)$	$\mathbb{P}(q_0, q_1, q_2, q_3)$	c	(m_0, m_1, m_2, m_3)
Z_{13}	$x^6 + xy^3 + z^2 + w^{18}$	$\mathbb{P}(1, 3, 5, 9)$	2	$(0, 1, -1, 0)$
$Z_{1,0}$	$x^5y + xy^3 + z^2 + w^{14}$	$\mathbb{P}(1, 2, 4, 7)$	2	$(0, 1, -1, 0)$
Z_{17}	$x^4z + xy^3 + z^2 + w^{12}$	$\mathbb{P}(2, 3, 7, 12)$	1	
$W_{1,0}$	$x^6 + y^2z + z^2 + w^{12}$	$\mathbb{P}(1, 2, 3, 6)$	2	$(0, 1, -1, 0)$
W_{17}	$x^5y + y^2z + z^2 + w^{10}$	$\mathbb{P}(2, 3, 5, 10)$	1	
$U_{1,0}$	$x^3y + y^2z + z^3 + w^9$	$\mathbb{P}(1, 2, 3, 3)$	2	$(0, 1, -1, 0)$
Q_{12}	$x^5 + y^3 + xz^2 + zw^9$	$\mathbb{P}(1, 3, 5, 6)$	2	$(1, 0, 0, -1)$
$Z_{1,0}$	$x^7 + xy^3 + z^2 + yw^{10}$	$\mathbb{P}(1, 2, 4, 7)$	3	$(1, 0, -1, 0)$
E_{20}	$x^{11} + y^3 + z^2 + xw^{12}$	$\mathbb{P}(5, 6, 22, 33)$	1	
$Q_{2,0}$	$x^4y + y^3 + xz^2 + zw^7$	$\mathbb{P}(1, 2, 4, 5)$	2	$(0, 1, -1, 0)$
Z_{18}	$x^6y + xy^3 + z^2 + yw^8$	$\mathbb{P}(3, 4, 10, 17)$	1	
E_{25}	$x^9y + y^3 + z^2 + xw^{10}$	$\mathbb{P}(5, 4, 18, 27)$	1	
Q_{16}	$x^4z + y^3 + xz^2 + zw^6$	$\mathbb{P}(2, 3, 7, 9)$	1	
$Z_{2,0}$	$x^5z + xy^3 + z^2 + yw^7$	$\mathbb{P}(1, 1, 3, 5)$	3	$(1, -1, -1, 1)$
E_{30}	$x^8z + y^3 + z^2 + xw^9$	$\mathbb{P}(5, 3, 16, 24)$	1	
$S_{1,0}$	$x^5 + xz^2 + y^2z + zw^6$	$\mathbb{P}(1, 2, 3, 4)$	2	$(0, 1, 0, -1)$
W_{18}	$x^7 + y^2z + z^2 + xw^8$	$\mathbb{P}(3, 4, 7, 14)$	1	
S_{16}	$x^4y + xz^2 + y^2z + zw^5$	$\mathbb{P}(2, 3, 5, 7)$	1	
$X_{2,0}$	$x^6y + y^2z + z^2 + xw^7$	$\mathbb{P}(1, 1, 3, 4)$	3	$(1, -1, 0, 0)$
U_{16}	$x^5 + y^2z + yz^2 + xw^6$	$\mathbb{P}(2, 3, 5, 5)$	1	

Table 4: Compactifications in weighted projective spaces

4 Configuration of rational curves on X

We want to study configurations of rational curves on X . Start by considering the curves

$$\begin{aligned}
C_\infty &:= (\{w = 0\} \cap Z) / \mathbb{Z}_c & \text{and} & & C_0 &:= (\{x = 0\} \cap Z) / \mathbb{Z}_c & \text{in } Y, \\
E_\infty &:= \pi^{-1}(C_\infty) & \text{and} & & E_0 &:= \pi^{-1}(C_0) & \text{in } X.
\end{aligned}$$

Proposition 2. *The curves C_0 and C_∞ are rational curves on Y .*

Proof. The curves $\{w = 0\} \cap Z$ and $\{x = 0\} \cap Z$ are quasismooth weighted complete intersections in $\mathbb{P}(Q)$ of multidegree (d, q_0) and (d, q_1) respectively. According to [D2, 3.4.4 Corollary], their genus is equal to zero except in the cases $Z_{2,0}$ and $X_{2,0}$ where it is equal to one. If the genus is already zero in Z , then also for the image curve in Y . For $Z_{2,0}$ and $X_{2,0}$, the form

$$\omega_1 := \frac{q_1 x dy dz - q_2 y dx dz + q_3 z dx dy}{df}$$

is a holomorphic 1-form on $\{w = 0\} \cap Z$ which generates the space of holomorphic 1-forms on this curve. However, it is not invariant with respect to the action of the group \mathbb{Z}_c . A similar argument holds for the curve $\{x = 0\} \cap Z$. \square

The surface Y has three cyclic quotient singularities of type $(\alpha_i, \alpha_i - 1)$ ($i = 1, 2, 3$) along the curve C_∞ . The curve C_0 intersects the curve C_∞ in some of these singularities. In order to compute how the curve E_0 meets the exceptional divisor of the resolution $\pi: X \rightarrow Y$, we study the local setting around a cyclic quotient singularity.

Local setting: We first consider \mathbb{C}^2 with the coordinates x, y and an action of the cyclic group \mathbb{Z}_k by $(x, y) \mapsto (\zeta x, \zeta^{-1} y)$ where ζ is a generator of \mathbb{Z}_k . The quotient $\mathbb{C}^2/\mathbb{Z}_k$ defines a cyclic quotient singularity of type $(k, k-1)$. It is well known that its resolution is obtained as follows: The polynomials x^k, y^k, xy are invariant under \mathbb{Z}_k . The map

$$\psi: \mathbb{C}^2 \rightarrow \mathbb{C}^3, \quad (x, y) \mapsto (X, Y, Z) = (x^k, y^k, xy)$$

factors through $\mathbb{C}^2/\mathbb{Z}_k$ and the image of the induced map is the hypersurface

$$\{(X, Y, Z) \in \mathbb{C}^3 \mid XY = Z^k\}.$$

The resolution $M \rightarrow \mathbb{C}^2/\mathbb{Z}_k$ is obtained by glueing k copies of \mathbb{C}^2 (with coordinates (u_i, v_i) , $i = 1, \dots, k$) by the maps

$$\phi_i: \mathbb{C}^2 \setminus \{v_i = 0\} \rightarrow \mathbb{C}^2 \setminus \{v_{i+1} = 0\}, \quad (u_i, v_i) \mapsto \left(\frac{1}{v_i}, u_i v_i^2\right) = (u_{i+1}, v_{i+1}).$$

Considering the singularity as a hypersurface, the resolution is given by the mapping $\pi_0: M \rightarrow \mathbb{C}^3$ in the coordinates (u_i, v_i) with

$$(u_i, v_i) \mapsto (X, Y, Z) = (u_i^i v_i^{i-1}, u_i^{k-i} v_i^{k+1-i}, u_i v_i).$$

The exceptional divisor is

$$E = \bigcup_{i=1}^{k-1} E_i, \quad E_i = \{u_i = v_{i+1} = 0\}, \quad i = 1, \dots, k-1.$$

We have $E_i \cap E_{i+1} \neq \emptyset$ for $i = 1, \dots, k-1$ and $E_i \cap E_j = \emptyset$ otherwise. The dual graph corresponding to the components E_i is a graph of type A_{k-1} . Note that the proper preimage of the curve $y = 0$ under the resolution $M \rightarrow \mathbb{C}^2/\mathbb{Z}_k$ intersects (transversally) the component E_1 of the exceptional divisor.

Lemma 3. *Let $0 < m < k$ be an integer. In \mathbb{C}^2 with coordinates x, y consider the curve $x^m + y^{k-m} = 0$. Then the proper preimage of this curve under the resolution $M \rightarrow \mathbb{C}^2/\mathbb{Z}_k$ intersects (transversally) the component E_{k-m} of the exceptional divisor.*

Proof. Under the map ψ , the curve $x^m + y^{k-m} = 0$ is mapped to the curve $Z^m + Y = 0$. In the coordinates (u_{k-m}, v_{k-m}) the preimage of this curve looks as follows:

$$u_{k-m}^m v_{k-m}^m + u_{k-m}^m v_{k-m}^{m+1} = u_{k-m}^m v_{k-m}^m (1 + v_{k-m}). \quad \square$$

Lemma 4. *In \mathbb{C}^2 with coordinates x, y consider the curve $x^2 + y^{2k-2} = 0$. Then the proper preimage of this curve under the resolution $M \rightarrow \mathbb{C}^2/\mathbb{Z}_k$ has two components which intersect (transversally) the component E_{k-1} of the exceptional divisor in two distinct points.*

Proof. Under the map ψ , the curve $x^2 + y^{2k-2} = 0$ is mapped to the curve $Z^2 + Y^2 = 0$. In the coordinates (u_{k-1}, v_{k-1}) the preimage of this curve looks as follows:

$$u_{k-1}^2 v_{k-1}^2 + u_{k-1}^2 v_{k-1}^4 = u_{k-1}^2 v_{k-1}^2 (1 + v_{k-1}^2). \quad \square$$

Application: We use these lemmas to compute the configurations of smooth rational curves on X . A smooth rational curve on a K3 surface has self-intersection number -2 by the adjunction formula. For the 6 quadrilateral singularities, all the singularities of Y lie on the curve C_∞ . For the 14 exceptional bimodal singularities, the surface Y has an additional singularity $P_0 = (1 : 0 : 0 : 0)$. This is a cyclic quotient singularity of type $(a, a-1)$ where a is defined in Section 2. It also lies on the curve C_0 . In the case $a = 5$, Lemma 3 implies that the curve E_0 intersects one of the inner components of the exceptional divisor corresponding to this singularity whose dual graph is of type A_4 . It turns out that the configurations of rational curves can be described with the help of Table 3 in a similar way as the Coxeter-Dynkin diagrams:

Proposition 5. *Let $f(x, y, z)$ be one of the invertible polynomials of Table 2 with invariants $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)$ and let Y be the surface constructed above. Then the total transform of the curve C_∞ under the resolution $\pi: X \rightarrow Y$ is a tree of smooth rational curves with the proper transform E_∞ as central curve and three branches of lengths $\alpha_1, \alpha_2, \alpha_3$.*

- (i) *If f^T defines a singularity of Kodaira type I_0^* with $r = 1$, the curve E_0 has two connected components E'_0 and E''_0 . These are smooth rational curves which intersect the outermost curve of the third branch and no other component of the exceptional divisor.*
- (ii) *Otherwise, the curve E_0 is smooth and rational (in particular, irreducible). If $\beta_i = \alpha_i - 1$, then the curve E_0 does not intersect any curve of the i -th branch. Otherwise, the curve E_0 intersects the $\alpha_i - \beta_i + 1$ -th outermost curve of the i -th branch. If P_0 lies on Y , then E_0 also intersects one component of the exceptional divisor of the resolution of this singularity.*

Proof. This is proved case by case using Lemma 3 and Lemma 4. We give some examples of this calculation.

Example $Z_{1,0}$. Here, $F(w, x, y, z) = x^5 y + xy^3 + z^2 + w^{14}$ and

$$Z := \{(w : x : y : z) \in \mathbb{P}(1, 2, 4, 7) \mid F(w, x, y, z) = 0\}.$$

We first consider the chart $U_1 := \{(w : x : y : z) \in \mathbb{P}(1, 2, 4, 7) \mid x = 1\}$. Then $U_1 = \mathbb{C}^3/\mathbb{Z}_2$ where \mathbb{Z}_2 acts on \mathbb{C}^3 by $(w, y, z) \mapsto (-w, y, -z)$. This action has 3 fixed points on

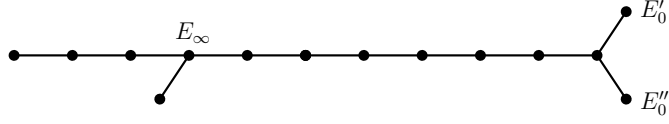


Figure 2: The configuration of rational curves in the case $Z_{1,0}$.

$Z_1 := \{(w, y, z) \in \mathbb{C}^3 \mid F(w, 1, y, z) = 0\}$, namely $P_1 = (0, \sqrt{-1}, 0)$, $P'_1 = (0, -\sqrt{-1}, 0)$, and $P_2 = (0, 0, 0)$.

Moreover, let $U_2 := \{(w : x : y : z) \in \mathbb{P}(1, 2, 4, 7) \mid y = 1\}$. Then $U_2 = \mathbb{C}^3/\mathbb{Z}_4$ where a generator $\zeta \in \mathbb{Z}_4$ acts on \mathbb{C}^3 by $(w, x, z) \mapsto (\zeta w, \zeta^2 x, \zeta^7 z)$. The only fixed point on $Z_2 := \{(w, x, z) \in \mathbb{C}^3 \mid F(w, x, 1, z) = 0\}$ is $P_3 = (0, 0, 0)$. The surface $Z_2 = \{x^5 + x + z^2 + w^{14} = 0\}$ is regular in x and the \mathbb{Z}_4 -action on the coordinates (w, z) is given by $(w, z) \mapsto (\zeta w, \zeta^{-1} z)$. Therefore the surface $Z \cap U_2$ has an A_3 singularity in P_3 .

We consider the action of the cyclic group \mathbb{Z}_2 on $\mathbb{P}(1, 2, 4, 7)$ given by

$$(w : x : y : z) \mapsto (w : -x : -y : z).$$

Under this action, the two points P_1 and P'_1 are identified, P_2 gets a cyclic quotient singularity of type $(4, 3)$, and P_3 becomes a cyclic quotient singularity of type $(8, 7)$. The curve $\{x = 0\}$ only meets the point P_3 . The singularity P_3 of $Y = Z/\mathbb{Z}_2$ is Z_2/\mathbb{Z}_8 . By Lemma 4, the proper preimage of the curve C_0 under the resolution $\pi: X \rightarrow Y$ consists of two components E'_0 and E''_0 which intersect (transversally) the component E_7 of the exceptional divisor $\pi^{-1}(P_3)$. Therefore we have the configuration depicted in Figure 2.

Example S_{16} . Here $F(w, x, y, z) = x^4 y + x z^2 + y^2 z + z w^5$ and

$$Y = Z := \{(w : x : y : z) \in \mathbb{P}(2, 3, 5, 7) \mid F(w, x, y, z) = 0\}.$$

This surface has 4 singularities, namely $P_0 = (1 : 0 : 0 : 0)$ of type A_1 , $P_1 = (0 : 1 : 0 : 0)$ of type A_2 , $P_2 = (0 : 0 : 1 : 0)$ of type A_4 , and finally $P_3 = (0 : 0 : 0 : 1)$ of type A_6 . The curve $\{x = 0\}$ goes through the points P_0 , P_2 , and P_3 . One can easily see that it intersects the curve $\{w = 0\}$ transversally in the point P_2 . To compute the intersection behaviour with the curve $\{w = 0\}$ at the point P_3 , consider the chart $U_3 := \{(w : x : y : z) \in \mathbb{P}(2, 3, 5, 7) \mid z = 1\}$. In this chart, Y is given by the equation $x^4 y + x + y^2 + w^5$. Therefore Lemma 3 implies that the curve E_0 intersects the component E_5 of the exceptional divisor of the A_6 singularity P_3 . Therefore we obtain the configuration depicted in Figure 3.

Example E_{20} . In this case $F(w, x, y, z) = x^{11} + y^3 + z^2 + x w^{12}$ and

$$Y = Z := \{(w : x : y : z) \in \mathbb{P}(5, 6, 22, 33) \mid F(w, x, y, z) = 0\}.$$

We first consider the chart $U_1 := \{(w : x : y : z) \in \mathbb{P}(5, 6, 22, 33) \mid x = 1\}$. Then $U_1 = \mathbb{C}^3/\mathbb{Z}_6$ where a generator $\zeta \in \mathbb{Z}_6$ acts on \mathbb{C}^3 by $(w, y, z) \mapsto (\zeta^5 w, \zeta^{22} y, \zeta^{66} z)$. The singularity $(0 : 1 : 0 : 0) \in U_1$ of the weighted projective space $\mathbb{P}(5, 6, 22, 33)$

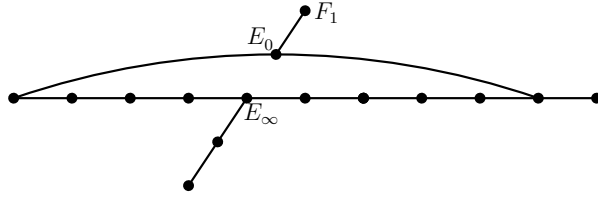


Figure 3: The configuration of rational curves in the case S_{16} .



Figure 4: The configuration of rational curves in the case E_{20} .

does not lie on Z , but the invariant surface $Z_1 := \{(w, y, z) \in \mathbb{C}^3 \mid F(w, 1, y, z) = 0\}$ has two points with non-trivial isotropy group, namely $P_1 = (0, -1, 0)$ with isotropy group of order 2 and $P_2 = (0, 0, \sqrt{-1})$ with isotropy group of order 3. They yield singularities of type A_1 and A_2 respectively. A similar reasoning for the chart $U_2 := \{(w : x : y : z) \in \mathbb{P}(5, 6, 22, 33) \mid y = 1\}$ shows that the surface Z has a third singularity $P_3 = (0 : 0 : 1 : \sqrt{-1})$ of type A_{10} . In this chart, Z is given by $Z_2 := \{(w, x, z) \in \mathbb{C}^3 \mid x^{11} + 1 + z^2 + xw^{12} = 0\}$. In local coordinates $(\xi_0, \xi_1, \xi_3) = (w, x, z - \sqrt{-1})$ around P_3 , where P_3 becomes the origin, the equation of Z_2 is given by $\xi_1^{11} + \xi_3^2 + 2\sqrt{-1}\xi_3 + \xi_1\xi_0^{12} = 0$. This shows that the curve $\{x = 0\}$ intersects the curve $\{w = 0\}$ in P_3 transversally. Therefore the proper preimages of these curves under the resolution $\pi : X \rightarrow Y$ intersect the first and the last component of the exceptional divisor $\pi^{-1}(P_3)$ respectively.

Now the surface Y has an additional singularity $P_0 = (1 : 0 : 0 : 0)$. Consider the corresponding chart $U_0 := \{(w : x : y : z) \in \mathbb{P}(5, 6, 22, 33) \mid w = 1\}$. Then $U_0 = \mathbb{C}^3/\mathbb{Z}_5$ where a generator $\zeta \in \mathbb{Z}_5$ acts on \mathbb{C}^3 by $(x, y, z) \mapsto (\zeta^6 x, \zeta^{22} y, \zeta^{33} z)$. Therefore P_0 is an A_4 singularity. The curve $\{x = 0\}$ in this chart is given by $y^3 + z^2$. It follows from Lemma 3 that the proper preimage of this curve under the resolution $\pi : X \rightarrow Y$ intersects the component E_2 of the exceptional divisor of $\pi^{-1}(P_0)$. Therefore we obtain the configuration depicted in Figure 4. \square

5 Categories and Coxeter-Dynkin diagrams

We have seen that the dual graphs of the curve configurations which we have constructed in the previous section are very similar to parts of the corresponding Coxeter-Dynkin diagrams of the bimodal singularities. We now want to realize the precise diagrams as Coxeter-Dynkin diagrams corresponding to certain sets of generators in triangulated categories associated to the above curve configurations. Let us note right away that the construction is geometric: we are providing a collection of sheaves on X . In the end, we

will come up with a category whose associated lattice from K-theory coincides with the Milnor lattice of the corresponding singularity.

All our categories will be built in the following way: Starting with a K3 surface X and a configuration of smooth rational -2 -curves, we will consider the smallest triangulated subcategory \mathcal{T} of the bounded derived category $D^b(X)$ (of coherent sheaves) which is generated by the structure sheaf \mathcal{O}_X and line bundles supported on -2 -curves. In certain cases, we have to apply a base change by way of a spherical twist.

Regarding the Ext groups of those sheaves, the relevant facts are collected in the following statement, where we make use of the complex $\mathrm{Hom}^\bullet(A, B) = \mathrm{Hom}(A, B) \oplus \mathrm{Ext}^1(A, B)[-1] \oplus \mathrm{Ext}^2(A, B)[-2]$ for sheaves A, B on X (this is a complex with zero differentials, so can be seen as a graded vector space).

Lemma 6. *Let X be a K3 surface and $C, D \subset X$ be two smooth rational -2 -curves. Then $\mathrm{Hom}^\bullet(\mathcal{O}_X, \mathcal{O}_C) = \mathbb{C}$, $\mathrm{Hom}^\bullet(\mathcal{O}_X, \mathcal{O}_C(-1)) = 0$, $\mathrm{Hom}^\bullet(\mathcal{O}_C, \mathcal{O}_C(-1)) = \mathbb{C}^2[-2]$. Furthermore, if C and D intersect transversally then $\mathrm{Hom}^\bullet(\mathcal{O}_C(i), \mathcal{O}_D(j)) = \mathbb{C}[-1]$ for any $i, j \in \mathbb{Z}$, whereas $\mathrm{Hom}^\bullet(\mathcal{O}_C(i), \mathcal{O}_D(j)) = 0$ if C and D are disjoint.*

Since the canonical bundle of X is trivial, the Serre functor of $D^b(X)$ is just the shift $[2]$, and the same is then true for \mathcal{T} . Such a category is often called a ‘2-Calabi-Yau category’. This implies that the Grothendieck K-group $K(\mathcal{T})$, equipped with the negative Euler pairing

$$-\chi([A], [B]) = -\sum_{i \in \mathbb{Z}} (-1)^i \dim \mathrm{Hom}_{\mathcal{T}}(A, B[i]),$$

is a lattice. What is more, \mathcal{T} will be generated by spherical objects, i.e. objects $S \in \mathcal{T}$ with $\mathrm{Hom}^\bullet(S, S) = \mathbb{C} \oplus \mathbb{C}[-2]$. Such objects give rise to roots $[S] \in K(\mathcal{T})$. The structure sheaf \mathcal{O}_X is spherical — this is just rephrasing the fact that X is a K3 surface. It is well-known that a line bundle on a chain of -2 -curves is spherical. And as is standard by now, a spherical object S gives rise to an autoequivalence T_S of the category, the *spherical twist* associated to S . Since \mathcal{T} is 2-Calabi-Yau, the autoequivalence T_S descends to the reflection of $(K(\mathcal{T}), -\chi(-, -))$ induced by the root $[S]$.

According to Proposition 5, the surface X comes with a star-like configuration of -2 -curves, given by $\pi^{-1}(C_\infty)$. This graph has three arms of lengths $\alpha_1, \alpha_2, \alpha_3$. We denote the corresponding curves by E_j^i where $i = 1, 2, 3$ and $j = 1, \dots, \alpha_i - 1$, starting at the outer ends. The central vertex corresponds to the curve E_∞ , it meets the curves $E_{\alpha_i-1}^i$. Furthermore, there is always the curve E_0 , as the strict transform of C_0 ; in three cases it decomposes into two components E_0' and E_0'' . For the 14 exceptional singularities, there are additional -2 -curves from resolving the cyclic quotient singularity P_0 ; we call them F_ℓ .

The situation is simplest for the singularities dual to the bimodal singularities with $a = 2$ except those of Kodaira type I_0^* with $r = 1$ ($J_{3,0}, Z_{1,0}, Q_{2,0}$). In this case, the curve configuration consists of the central curve E_∞ , the three arms E_j^i and the additional curve E_0 . In the case of the exceptional bimodal singularities, we have an additional curve F_1 coming from the singularity P_0 . This will not be used. We define the

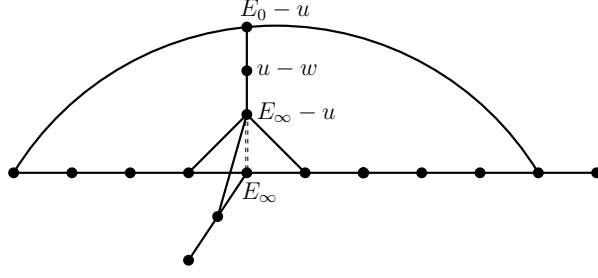


Figure 5: Coxeter-Dynkin diagram for S_{16} .

category \mathcal{T} as the smallest triangulated subcategory of $D^b(X)$ containing the following objects:

Case $a = 2$ ($W_{1,0}, S_{1,0}, U_{1,0}, E_{18}, Z_{17}, Q_{16}, W_{17}, S_{16}, U_{16}$)

$$\mathcal{T} = \langle \mathcal{O}_{E_1^1}(-1), \dots, \mathcal{O}_{E_{\alpha_1-1}^1}(-1), \mathcal{O}_{E_1^2}(-1), \dots, \mathcal{O}_{E_{\alpha_2-1}^2}(-1), \\ \mathcal{O}_{E_1^3}(-1), \dots, \mathcal{O}_{E_{\alpha_3-1}^3}(-1), \mathcal{O}_{E_\infty}(-1), \mathcal{O}_{E_\infty}, \mathcal{O}_X, \mathcal{O}_{E_0} \rangle$$

The K-group of this category is the lattice spanned by the curves of $\pi^{-1}(C_\infty)$ and E_0 , extended by a hyperbolic plane with a basis of isotropic elements u and w ; we use the roots $E_\infty - u$ and $u - w$ as generators. See Figure 5 for the singularity S_{16} . The lattice $K(\mathcal{T})$ can be seen as a sublattice of the cohomology $H^*(X, \mathbb{Z})$ equipped with the Mukai pairing. In this picture, $-u$ is the class of a skyscraper sheaf (of length 1) on the curve E_∞ , i.e. an element of $H^4(X, \mathbb{Z})$. The isotropic element $-w$ corresponds to the ideal sheaf of this point, so that $\text{ch}(\mathcal{O}_X) = u - w$. The correspondence between sheaves and lattice elements is furnished by the Chern character $\text{ch}: \mathcal{T} \hookrightarrow D^b(X) \rightarrow H^*(X, \mathbb{Z})$. Then $\text{ch}(\mathcal{O}_{E_\infty}(-1)) = E_\infty$ in $H^2(X, \mathbb{Z})$ and similar for the other curves. Furthermore, $\text{ch}(\mathcal{O}_{E_\infty}) = E_\infty - u$. For details see [EP1].

In the case $a = 3$, we use in addition the curve F_1 of the exceptional divisor of the A_2 singularity P_0 , but not F_2 :

Case $a = 3$ ($E_{19}, Z_{18}, Q_{17}, W_{18}, S_{17}$)

$$\mathcal{T} = \langle \mathcal{O}_{E_1^1}(-1), \dots, \mathcal{O}_{E_{\alpha_1-1}^1}(-1), \mathcal{O}_{E_1^2}(-1), \dots, \mathcal{O}_{E_{\alpha_2-1}^2}(-1), \\ \mathcal{O}_{E_1^3}(-1), \dots, \mathcal{O}_{E_{\alpha_3-1}^3}(-1), \mathcal{O}_{E_\infty}(-1), \mathcal{O}_{E_\infty}, \mathcal{O}_X, \mathcal{O}_{F_1}(-1), \mathcal{O}_{E_0} \rangle$$

The remaining cases are the singularities dual to the bimodal singularities with $a = 2$ of Kodaira type I_0^* with $r = 1$ ($J_{3,0}, Z_{1,0}, Q_{2,0}$) and the three singularities with $a = 5$ of Kodaira type II^* . In each case, there is one curve inside the third branch of the curve configuration which we have to omit.

To this end, we will apply a suitable base change. Let us denote the superfluous curve momentarily by B . For the sake of simplicity, we assume that B is incident to just two other smooth rational curves A and C . The base change we are after is $[C] \mapsto [B] + [C]$;

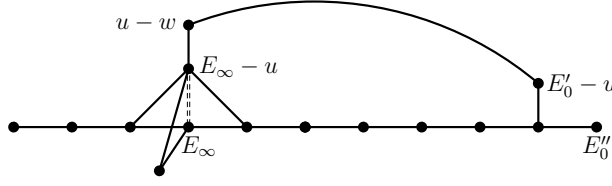
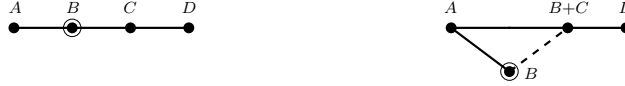


Figure 6: Coxeter-Dynkin diagram for $Z_{1,0}$.

note that this is the reflection along the root B applied to C . Omitting the curve B leaves us with a chain one vertex shorter, as desired:



On the categorical level, we use that the spherical twist $\mathsf{T}_{\mathcal{O}_B(-1)}$ is a lift of the reflection, i.e. we use the sheaf $\mathsf{T}_{\mathcal{O}_B(-1)}(\mathcal{O}_C(-1))$ which is defined by the short exact sequence $0 \rightarrow \mathcal{O}_C(-1) \rightarrow \mathsf{T}_{\mathcal{O}_B(-1)}(\mathcal{O}_C(-1)) \rightarrow \mathcal{O}_B(-1) \rightarrow 0$. This non-split extension is unique as a sheaf and a line bundle supported on $B \cup C$. It follows immediately from this sequence and Lemma 6 that the Hom^\bullet -groups are preserved. As a consequence of this, the intersection behaviour, given by the negative of the Euler form on the category, is unchanged.

We define the category \mathcal{T} as the smallest triangulated subcategory of $D^b(X)$ containing the following objects:

Case $a = 2$ ($J_{3,0}, Z_{1,0}, Q_{2,0}$)

$$\begin{aligned} \mathcal{T} = \langle & \mathcal{O}_{E_1^1}(-1), \dots, \mathcal{O}_{E_{\alpha_1-1}^1}(-1), \mathcal{O}_{E_1^2}(-1), \dots, \mathcal{O}_{E_{\alpha_2-1}^2}(-1), \\ & \mathsf{T}_{\mathcal{O}_{E_1^3}(-1)}(\mathcal{O}_{E_2^3}(-1)), \mathcal{O}_{E_3^3}(-1), \dots, \mathcal{O}_{E_{\alpha_3-1}^3}(-1), \\ & \mathcal{O}_{E_{\infty}}(-1), \mathcal{O}_{E_{\infty}}, \mathcal{O}_X, \mathcal{O}_{E_0} \rangle \end{aligned}$$

Case $a = 5$ (E_{20}, Z_{19}, Q_{18})

$$\begin{aligned} \mathcal{T} = \langle & \mathcal{O}_{E_1^1}(-1), \dots, \mathcal{O}_{E_{\alpha_1-1}^1}(-1), \mathcal{O}_{E_1^2}(-1), \dots, \mathcal{O}_{E_{\alpha_2-1}^2}(-1), \\ & \mathsf{T}_{\mathcal{O}_{E_1^3}(-1)}(\mathcal{O}_{E_2^3}(-1)), \mathcal{O}_{E_3^3}(-1), \dots, \mathcal{O}_{E_{\alpha_3-1}^3}(-1), \\ & \mathcal{O}_{E_{\infty}}(-1), \mathcal{O}_{E_{\infty}}, \mathcal{O}_X[1], \mathcal{O}_{F_1}, \mathcal{O}_{F_2}(-1), \mathcal{O}_{F_3}(-1), \mathcal{O}_{F_4}(-1), \mathcal{O}_{E_0}(-1) \rangle. \end{aligned}$$

The Coxeter-Dynkin diagrams corresponding to these sets of generators for the singularities $Z_{1,0}$ and E_{20} are depicted in Figure 6 and Figure 7 respectively.

Summarising, we obtain the following theorem.

Theorem 7. *Let \mathcal{T} be one of the triangulated categories associated above with a bimodal singularity. Then the lattice $K(\mathcal{T})$, equipped with the negative Euler pairing, is isomorphic to the Milnor lattice of the singularity and the Coxeter-Dynkin diagram corresponding to the above system of generators of \mathcal{T} coincides with the Coxeter-Dynkin diagram corresponding to a distinguished basis of vanishing cycles of the singularity.*

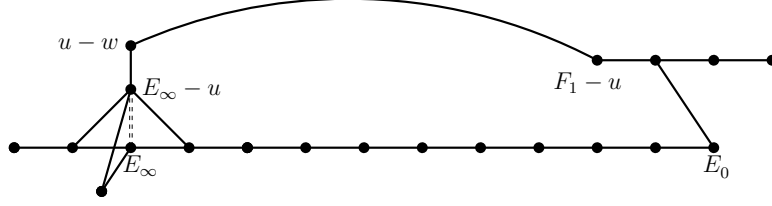


Figure 7: Coxeter-Dynkin diagram for E_{20} .

6 Coxeter elements

Let \mathcal{T} be one of the above categories. A spherical object D in \mathcal{T} gives rise to a spherical twist whose action on $(K(\mathcal{T}), -\chi(-, -))$ is just the reflection $s_{[D]}$ along the class $[D] \in K(\mathcal{T})$.

Corollary 8. *Let \mathcal{T} be one of the triangulated categories associated above with a bimodal singularity. The Coxeter element corresponding to the above system of generators of \mathcal{T} corresponds to the monodromy operator of the singularity.*

Remark 9. Since the triangulated category \mathcal{T} is generated by 2-spherical objects, there is a Coxeter functor, given by composing all the spherical twists of the spherical objects comprising the basis of \mathcal{T} . This functor lifts the Coxeter element from an isometry of the lattice to an autoequivalence of \mathcal{T} .

If τ is the monodromy operator, then we consider the polynomial $\Delta(t) = \det(1 - \tau^{-1}t)$ as its characteristic polynomial, using a suitable normalization.

Let $f(x, y, z)$ be a non-degenerate invertible polynomial and let $(w_1, w_2, w_3; d')$ be the canonical weight system corresponding to $f(x, y, z)$. The ring $R_f := \mathbb{C}[x, y, z]/(f)$ is a \mathbb{Z} -graded ring. Therefore, we can consider the decomposition of R_f as a \mathbb{Z} -graded \mathbb{C} -vector space:

$$R_f := \bigoplus_{k \in \mathbb{Z}_{\geq 0}} R_{f,k}, \quad R_{f,k} := \left\{ g \in R_f \mid w_1 x \frac{\partial g}{\partial x} + w_2 y \frac{\partial g}{\partial y} + w_3 z \frac{\partial g}{\partial z} = kg \right\}.$$

The formal power series

$$p_f(t) := \sum_{k \geq 0} (\dim_{\mathbb{C}} R_{f,k}) t^k \tag{1}$$

is the *Poincaré series* of the \mathbb{Z} -graded coordinate ring R_f with respect to the canonical system of weights $(w_1, w_2, w_3; d')$ attached to f . It is given by

$$p_f(t) = \frac{(1 - t^{d'})}{(1 - t^{w_1})(1 - t^{w_2})(1 - t^{w_3})}.$$

Let $(\alpha_1, \alpha_2, \alpha_3)$ be the Dolgachev numbers of f (see [ET]). Consider the polynomial

$$\Delta_0(t) = (1 - t)^{-2}(1 - t^{\alpha_1})(1 - t^{\alpha_2})(1 - t^{\alpha_3}).$$

The rational function

$$\phi_f(t) := p_f(t)\Delta_0(t)$$

is called the *characteristic function of f* . From Table 10 and Table 11 of [ET] we can derive the following theorem.

Theorem 10. *Let $f(x, y, z)$ be a non-degenerate invertible polynomial and assume that the canonical system of weights attached to f^T is reduced. Then $\phi_f(t)$ is the characteristic polynomial of the monodromy operator of f^T .*

If f^T is the invertible polynomial in Table 2 corresponding to one of the 14 exceptional bimodal singularities, then its canonical system of weights is reduced. Therefore we can apply Theorem 10 in these cases. Note that in these cases $\Delta_0(t)$ is the characteristic polynomial of the Coxeter element corresponding to the subset of generators of \mathcal{T} with support on the preimage of C_∞ under the resolution $\pi: X \rightarrow Y$ with the Coxeter-Dynkin diagram given by Figure 1. Then we get the following corollary of Theorem 10:

Corollary 11. *Let $f(x, y, z)$ be an invertible polynomial which is the Berglund-Hübsch transpose of an invertible polynomial with a reduced canonical system of weights defining an exceptional bimodal singularity. Then*

$$p_f(t) = \frac{\Delta(t)}{\Delta_0(t)}$$

where $\Delta(t)$ is the characteristic polynomial of the Coxeter element corresponding to the above system of generators of \mathcal{T} .

A similar result holds for Fuchsian singularities [EP1, EP2]. There we gave a geometric proof of this fact. It is an open problem to derive a similar proof for Corollary 11.

Remark 12. Since the canonical systems of weights of Z_{17} and W_{17} are reduced, we can apply [ET, Theorem 22] and obtain that $\phi_f(t)$ is the characteristic polynomial of an operator τ such that τ^2 is the Coxeter element corresponding to the above system of generators of \mathcal{T} . It can be checked that a similar result holds for $U_{1,0}$. For the remaining quadrilateral singularities, there is no such relation between $\phi_f(t)$ and the Coxeter element.

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